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Input-to-state stability of nonlinear systems in terms of two measures

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1 Introduction

The notion of input-to-state stability (ISS) has been proven powerful in stability analysis and control synthesis for dynamical systems. This notion characterizes the effects of external inputs on the stability of control systems. In the stability notion, if we generalize the Euclidean norm to functions in Gamma, we will. On the other hand, the notion of stability in terms of two measures allows us to study Lyapunov stability, partial stability, orbit stability, and stability of the invariant set for nonlinear systems simultaneously. But the literature lacks a formal definition and criteria for ISS in terms of two measures. Therefore, we fill in the gap in the literature by developing ISS in terms of two measures. For the convenience of real-life applications, we extend ISS in terms of two measures to systems on time scale.

2 Lyapunov analysis of nonlinear systems

2.1 Premilinaries

Definition 2.1.1. Given two metric spaces $(X,d_X),(Y,d_Y)$, a function $f:X\to Y$ is called Lipschitz if there exists a real constant $K\geqslant 0$ such that, for all x_1 and x_2 in X,

$$d_Y(f(x_1)-f(x_2))\leqslant Kd_X(x_1,x_2) \tag{1}$$

f is called locally Lipschitz if for every $x\in X$, there exists a neighbourhood U of x such that f restricted to U is Lipschitz.

Definition 2.1.2. A function f is positive definite if and only if $f(0)=0$ and $f(x)>0$ for $x\neq 0$

Definition 2.1.3. A function f is positive semidefinite if and only if $f(0)=0$ and $f(x)\geqslant 0$ for $x\neq 0$

Definition 2.1.4. A function f is negative definite if and only if $f(0)=0$ and $-f(x)>0$ for $x\neq 0$

Definition 2.1.5. A function f is negative semidefinite if and only if $f(0)=0$ and $-f(x)\geqslant 0$ for $x\neq 0$

Definition 2.1.6. A continous function $\alpha:[0,a)\to[0,\infty)$ is said to belong to class $\mathscr{K}$ if and only if it is strictly increasing and $\alpha(0)=0$. It is said to belong to class $\mathscr{K}_\infty$ if $a=\infty$ and $\alpha(r)\to\infty$ as $r\to\infty$.

Definition 2.1.7. A contuinous function $\alpha:[0,a)\to[0,\infty)$ is said to belong to class $\mathscr{L}$ if and only if it is strictly decreasing and $\lim_{x\to\infty}\alpha(x)=0$.

Definition 2.1.8. A continuous function $\beta:[0,\infty)\times[0,\infty)\to[0,\infty)$ is said to belong to class $\mathscr{KL}$ if and only if

- for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathscr{K}$ with respect to r

- for each fixed r, the mapping $\beta(r,s)$ belongs to class $\mathscr{L}$ with respect to s

Definition 2.1.9. A continuous function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to belong to class $\mathscr{CK}$ if and only if for each fixed r, the mapping $\beta(r,s)$ belongs to class $\mathscr{K}$ with respect to s.

Lemma 2.1.1.
- *The composition of class $\mathscr{K}$ functions belongs to class $\mathscr{K}$*
- *The composition of class $\mathscr{K}_\infty$ functions belongs to class $\mathscr{K}_\infty$*
- *$\sigma(r,s) = \alpha_2(\beta(\alpha_1(r)),s)$ belongs to class $\mathscr{KL}$, where α_1, α_2 belongs to class $\mathscr{K}$, β belongs to class $\mathscr{KL}$.*

Lemma 2.1.2. *Let $V : D \to \mathbb{R}$ be a continuous positive definite function defined on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathscr{K}$ functions α_1, α_2 defined on $[0,r]$, such that*

$$\alpha_1(\|x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|) \tag{2}$$

for all $x \in B_r$. If $D = \mathbb{R}^n$, the functions α_1, α_2 will be defined on $[0,\infty)$ and the forgoing inequality will hold for all $x \in \mathbb{R}^n$. Moreover, if $V(x)$ is radially unbounded, then α_1, α_2 can be chosen to belong to class $\mathscr{K}_\infty$.

Proof. Consider

$$\phi(s) = inf_{s \leqslant \|x\| \leqslant r} V(x) \tag{3}$$

for $0 \leqslant s \leqslant r$.

$$\psi(s) = sup_{\|x\| \leqslant s} V(x) \tag{4}$$

for $0 \leqslant s \leqslant r$. $\phi(\cdot)$ and $\psi(\cdot)$ are both continuous, positive definite and nondecreasing. Moreover,

$$\phi(\|x\|) \leqslant V(x) \leqslant \psi(\|x\|) \tag{5}$$

There exists class $\mathscr{K}$ functions α_1, α_2 such that

$$\alpha_1(\|x\|) \leqslant \phi(\|x\|), \psi(\|x\|) \leqslant \alpha_2(\|x\|) \tag{6}$$

Hence, $\alpha_1(\|x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|)$. If $D = \mathbb{R}^n$, let

$$\phi(s) = inf_{s \leqslant \|x\|} V(x), \psi(s) = sup_{\|x\| \leqslant s} V(x) \tag{7}$$

for all $x \in \mathbb{R}^n$, and the rest of the proof is the same. If $V(x)$ is radially unbounded, then $\phi(s), \psi(s) \to \infty$ as $s \to \infty$, which means α_1, α_2 needs to be chosen in class $\mathscr{K}_\infty$. □

Lemma 2.1.3. *Consider the scalar differential equation*

$$\dot{u} = f(t,u), u(t_0) = u_0 \tag{8}$$

where $f(t,u)$ is continuous in t and locally Lipschitz in u, for all $t \geqslant 0$ and all $u \in J \subset \mathbb{R}$. Let $[t_0, T)$ (T could be infinite) be the maximal interval of existence of the solution $u(t)$, and suppose $u(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a continuous function whose upper right-hand derivative $D^+v(t)$ satisfies the differential inequality

$$D^+v(t) \leqslant f(t,v(t)), v(t_0) \leqslant u_0 \tag{9}$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leqslant u(t)$ for all $t \in [t_0, T)$.

Proof. Consider the perturbed differential equation

$$\dot{z} = f(t,z) + \lambda, z(t_0) = u_0 \tag{10}$$

where $\lambda > 0$. On any compact interval $[t_0, t_0]$, for any $\varepsilon > 0$, there is $\delta > 0$ such that if $\lambda < \delta$, then Eq.10 has a unique solution $z(t,\lambda)$ defined on $[t_0, t_1]$ and $|z(t,\lambda) - u(t)| < \varepsilon, \forall t \in [t_0, t_1]$.
Claim: $v(t) \leqslant z(t,\lambda), \forall t \in [t_0, t_1]$
Proof: For the sake of contradiction, suppose there exists $a, b \in [t_0, t_1]$ such that $v(a) = z(a,\lambda)$ and $v(t) > z(t,\lambda), \forall t \in (a,b]$ ($v(t) > z(t,\lambda)$ over an interval because of the continuity of v). Therefore, we have

$$v(t) - v(a) > z(t,\lambda) - z(a,\lambda), \forall t \in (a,b] \tag{11}$$

which means

$$\frac{v(a+\delta) - v(a)}{\delta} > \frac{z(a+\delta,\lambda) - z(a,\lambda)}{\delta}, \forall t \in (a,b] \tag{12}$$

for all $\delta \in (0, b-a]$. By the property of limit supremum,

$$\limsup_{\delta \to 0} \frac{v(a+\delta) - v(a)}{\delta} > \limsup_{\delta \to 0} \frac{z(a+\delta,\lambda) - z(a,\lambda)}{\delta}, \forall t \in (a,b] \tag{13}$$

which means

$$D^+ v(a) > \dot{z}(a,\lambda) = f(t,a) + \lambda, \forall t \in (a,b] \tag{14}$$

which contradicts the assumption that $D^+ v(a) > \dot{z}(a,\lambda) < f(t,a), \forall t \in [t_0, T)$.
Claim: $v(t) \leqslant u(t), \forall t \in [t_0, t_1]$
Proof: For the sake of contradiction, suppose there exists $a \in (t_0, t_1]$ such that $v(a) > u(a)$. Take $\varepsilon = (v(a) - u(a))/2$, choose λ such that $|z(t,\lambda) - u(t)| < \varepsilon$. Hence $v(a) - z(a,\lambda) = v(a) - u(a) + u(a) - z(a,\lambda) \geqslant 2\varepsilon - \varepsilon = \varepsilon$, which means $v(a) \geqslant z(a,\lambda) + \varepsilon > z(a,\lambda)$. This contradicts the first claim above.
Hence, we showed that $v(t) \leqslant u(t)$ over any compact set $[t_0, t_1]$. We want to show it is true for all $t \geqslant t_0$. For the sake of contradiction, suppose T is the first time that it violates $v(t) \leqslant u(t)$, meaning $v(T) = u(T)$ by continuity of v. Since $v(t) \leqslant u(t)$ over any compact set, meaning the inequality is satisfied over $[T, T+\Delta]$ for some $\Delta > 0$, which means the first time violating $v(t) \leqslant u(t)$ has to be after $T + \Delta$, which is a contradiction.

□

Note that Lemma.2.1.3 has a Dini derivative version, which is important when generalizing the input-to-state stability in terms of two measures.

Lemma 2.1.4. *Consider the scalar autonomous differential equation*

$$\dot{y} = -\alpha(y), y(t_0) = y_0 \tag{15}$$

where α is a locally Lipschitz class $\mathscr{K}$ function defined on $[0,a)$. For all $0 \leqslant y_0 \leqslant a$, this equation has a unique solution $y(t)$ defined for all $t \geqslant t_0$. (Since $\dot{y} \leqslant 0$ and $y(t_0) \in [0,a]$, $y(t) \in [0,a]$ as long as $y_0 \in [0,a]$). Moreover,

$$y(t) = \sigma(y_0, t - t_0) \tag{16}$$

where σ is a class $\mathscr{K}\mathscr{L}$ function defined on $[0,a) \times [0,\infty)$.

Proof. Since α is locally Lipschitz, the differential equation has a unique solution for every initial state $y_0 \geqslant 0$. Since the differential equation is separable, when $y_0 \neq 0$:

$$-\int_{y_0}^{y} \frac{dx}{\alpha(x)} = \int_{t_0}^{t} d\tau = t - t_0 \tag{17}$$

Let $0 \geqslant b < a$ be any positive number less than a, and define

$$\eta(y) = -\int_{b}^{y} \frac{dx}{\alpha(x)} \tag{18}$$

Therefore, we have

$$\eta(y) - \eta(y_0) = t - t_0 \tag{19}$$

$$y(t) = \eta^{-1}(\eta(y_0) + t - t_0) \tag{20}$$

If $y_0 = 0$, $y(t) = 0$, hence

$$y(t) = \sigma(r,s) = \begin{cases} \eta^{-1}(\eta(r)+s), \; r > 0 \\ 0, \; r = 0 \end{cases} \tag{21}$$

STS $\sigma(r,s)$ is in class $\mathscr{KL}$. Note that $\eta(y)$ is continuous and strictly decreasing, η^{-1} is continuous. Note that as $t \to \infty$, $y(t) \to 0$. This can only happen asymptotically, since if $y(t)$ hits zero in finite time, then it would result2 in a zero derivative at that point, which would violate the uniqueness of the solution. Hence, $\lim_{x\to 0} \eta(x) = \infty$, $\lim_{x\to\infty} \eta^{-1}(x) = 0$. Therefore, $\sigma(r,s)$ is continuous.

$$\frac{\partial}{\partial r}\sigma(r,s) = \frac{\alpha(\sigma(r,s))}{\alpha(r)} > 0 \tag{22}$$

$$\frac{\partial}{\partial s}\sigma(r,s) = -\alpha(\sigma(r,s)) < 0 \tag{23}$$

Hence, $\sigma(r,s)$ is strictly increasing along r and strictly decreasing along s. Hence, $\sigma(r,s)$ belongs to class $\mathscr{KL}$.

□

2.2 Autonomous system

Suppose $f : D \to \mathbb{R}^n, D \subset \mathbb{R}^n$ is locally Lipschitz, consider the autonomous system

$$\dot{x} = f(x) \tag{24}$$

where f is not t dependent, meaning the solution only depends on $t - t_0$ where t_0 is the initial condition. All equilibrium points can be shifted to the origin with simple transformation $y = x - \bar{x}$ where $\bar{x}$ is the equilibrium. Hence, WLOG we assume the equilibrium is $x = 0$.

Definition 2.2.1. The equilibrium point $x = 0$ of Eq.24 is

- (Lyapunov) stable if $\forall\, \varepsilon > 0, \exists\, \delta(\varepsilon) > 0 x(0) \leqslant \delta,\; \|x(t)\| \leqslant \varepsilon,\; \forall t \geqslant 0$.

- unstable if it is not stable
- asymptotically stable if it is stable and δ can be chosen such that $\|x(0)\| \leqslant \delta \implies \lim_{t\to\infty} x(t) = 0$

Lyapunov stability only requires that as long as the solution starts close enough to the equilibrium, it will stay close. However, the asymptotic stability requires the solution to converge to the equilibrium.

Theorem 2.2.1. *Suppose $x = 0$ is the equilibrium of Eq.24, the equilibrium is stable if there exists a positive definite function $V : D \to \mathbb{R}, D \subset \mathbb{R}^n$ such that $\dot{V}$ is negative semidefinite. And it is asymptotically stable if $\dot{V}$ is negative definite.*

Proof. Given $\varepsilon > 0$. Consider $r \in (0, \varepsilon]$, $B_r = \{x | \|x\| \leqslant r\} \subset D$. Take a set in the interior of B_r as follows:
Let α be the minimum value of V on the boundary of B_r: $\alpha = \min_{\|x\|=r} V(x)$. Take $\beta \in (0, \alpha)$. Consider the set

$$\Omega_\beta = \{x \in B_r | V(x) \leqslant \beta\} \tag{25}$$

Ω_β is guaranteed to be in the interior of B_r by the definition of α. Note that all trajectories started in Ω_β will stay in there because $\dot{V}(x) < 0$.
Since Ω_β is closed(complement is open by continuity of V) and bounded (since it is the interior of B_r), it is compact, hence every solution started in Ω_β is unique.
By the continuity of $V(x)$ and $V(0) = 0$, there exists $\delta > 0$ such that if $\|x - 0\| < \delta$, $V(x) - V(0) = V(x) < \beta$. From here we deduce that if $\|x(0)\| \leqslant \delta$, then $\|x(t)\| \in \Omega_\beta \subset B_r$, meaning $\|x(t)\| < r \leqslant \varepsilon$, which shows the equilibrium is stable.
Now suppose also $\dot{V}$ is negative definite. We need to show the trajectory converges to the equilibrium, meaning for any $a > 0$, there exists $T > 0$ such that if $t \geqslant T$, then $\|x\| \leqslant a$. Since from the proof above, for any B_a, we can construct $\Omega_b \subset B_a$, which means STS $V(x) \to 0$. Since $V(x)$ is monotonically decreasing and bounded from below by zero, we can conclude that $V(x) \to c$ where $c \geqslant 0$. For the sake of contradiction, suppose $c > 0$. By continuity of $V(x)$, there is $d > 0$ such that $B_d \subset \Omega_c$. Since $V(x)$ approaches $c > 0$ from above, the trajectory lies outside Ω_c hence outside B_d for all $t \geqslant 0$. Let $-\gamma = max_{d \leqslant \|x\| \leqslant r} \dot{V}(x)$, which exists because the continuous function $\dot{V}(x)$ has a maximum over the compact set $\{d \leqslant \|x\| \leqslant r\}$. Since $\dot{V}$ is negative definite, $-\gamma < 0$. It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \leqslant V(x(0)) - \gamma t \tag{26}$$

The right-hand side can eventually become negative as t increases, which contradicts the fact that $c > 0$. Hence, $c = 0$, which shows the equilibrium is asymptotically stable. □

The Lyapunov function $V : D \to \mathbb{R}$ gives a sufficient condition for determining if the equilibrium is stable or not. Meaning if such a function doesn't exist, we cannot say the equilibrium is unstable. Also, since this is an existence theorem, finding the “right" Lyapunov function is crucial.
A question to ask about asymptotic stability is how far away from the equilibrium the system can start to still converge to the equilibrium, which is what is the region of attraction. Next, we study the case when the region of attraction is the whole space, which is globally asymptotically stable.

Theorem 2.2.2. *Let $x = 0$ be an equilibrium point for Eq.24. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0, and\, V(x) > 0\ \forall x \neq 0 \tag{27}$$

$$\|x\| \to \infty \implies V(x) \to \infty \tag{28}$$

$$\dot{V}(x) < 0, \forall x \neq 0 \tag{29}$$

Then $x = 0$ is globally asymptotically stable.

Proof. Given any $p \in \mathbb{R}^n$, let $c = V(p)$. The second condition implies for any $c > 0$, there exists $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Following the proof of Theorem. 2.2.1, consider Ω_c, which in this case $\Omega_c \subset B_r$, which implies Ω_c is bounded. The rest of the proof is similar to the proof of Theorem.2.2.1. □

The globally asymptotically stable equilibrium has to be the unique equilibrium. Suppose there is another equilibrium $\bar{x}$, the trajectories stars at $\bar{x}$ will stay at $\bar{x}$ and not converge to the origin, which contradicts the fact that the stability of origin is global. Hence, systems with multiple equilibria must not have global stability.

2.3 Nonautonomous system

Suppose $f : [0,\infty) \times D \to \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ containing $x = 0$ is piecewise continuous in t and locally Lipschitz in x. Consider the nonautonomous system

$$\dot{x} = f(t,x) \tag{30}$$

Different from an autonomous system, the solution of a nonautonomous system depends on both t and t_0. An equilibrium at the origin could be a translation of a non-zero solution to the system like what we did in the last section, or a translation of a non-zero equilibrium point. Therefore, WLOG we assume the equilibrium is at the origin. The definition of stability is different due to the t dependence in the system. The main difference lies in the notion of “uniformity" related to the t dependence when the chosen constant is time-independent.

Definition 2.3.1. The equilibrium point $x = 0$ of Eq.30 is

- stable if $\forall\ \varepsilon > 0, \exists\ \delta(t_0, \varepsilon) > 0$

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \varepsilon, \forall\ t \geqslant t_0 \geqslant 0 \tag{31}$$

- uniformly stable if $\forall \varepsilon > 0, \exists\ \delta(\varepsilon) > 0\, Eq.31 is satisfied.$
- unstable if not stable
- asymptotically stable if it is stable and there is a constant $c = c(t_0) > 0$ such that $x(t) \to 0$ as $t \to \infty, \forall\ \|x(t_0)\| < c$.
- uniformly asymptotically stable if it is uniformly stable and there is a constant $c > 0$(independent of t_0) such that $x(t) \to 0$ as $t \to \infty, \forall\ \|x(t_0)\| < c$.

- globally uniformly asymptotically stable if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim_{\varepsilon\to\infty}\delta(\varepsilon)=\infty$, and for each pair of positive numbers η and c there is $T=T(\eta,c)>0$ such that

$$\|x(t)\| < \eta, \forall t \geqslant t_0 + T(\eta,c), \forall \|x(t_0)\| < c \tag{32}$$

There are lemmas that establish more applicable definitions for uniform stability using the properties of class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

Lemma 2.3.1. *The equilibrium x=0 of Eq.30 is:*

- *uniformly stable if and only if there exists a class $\mathcal{K}$ function α and a constant $c>0$ (independent of t_0) such that*

$$\|x(t)\| \leqslant \alpha(\|x(t_0)\|) \tag{33}$$

  $\forall t \geqslant t_0 \geqslant 0, \forall \|x(t_0)\| < c$

- *uniformly asymptotically stable if and only if there exists a class $\mathcal{KL}$ function β and a constant $c>0$ (independent of t_0) such that*

$$\|x(t)\| \leqslant \beta(\|x(t_0)\|, t-t_0) \tag{34}$$

  $\forall t \geqslant t_0 \geqslant 0, \forall \|x(t_0)\| < c$

- *globally uniformly asymptotically stable if and only if Eq.34 is satisfies for any $x(t_0)$.*

Next, we extend Lyapunov's theory for autonomous systems to nonautonomous systems.

Theorem 2.3.1. *(uniform stability) If there exists x=0 be an equilibrium for Eq.30. Suppose $D\subset\mathbb{R}^n$ is a domain containing $x=0$. Let $V:[0,\infty)\times D\to\mathbb{R}$ be a continuously differentiable function such that*

$$W_1(x) \leqslant V(t,x) \leqslant W_2(x) \tag{35}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leqslant 0 \text{ (derivative along the trajectory of Eq.30)} \tag{36}$$

$\forall t \geqslant 0, \forall x \in D$, *where $W_1(x), W_2(x)$ are continuous positive definite functions on D. THEN, $x=0$ is uniformly stable.*

Note that $V(t,x)$ that satisfies Eq.35 has to be positive definite, which is analogous to the requirement of V being positive definite in the autonomous case. Eq.36 is the same as requiring a negative semidefinite derivative of V in the autonomous case, but now the path that we are taking derivative along is time dependent.

Proof. The proof consists of two parts: 1. show there exists $c>0$ such that the solution started in B_c remains in B_c. This can be shown using the same idea of proof of Theorem.2.2.1 and Eq.35. 2. show there exists class $\mathcal{K}$ function α such that $\forall \|x(t_0)\| \leqslant c, \|x(t)\| \leqslant \alpha(\|x(t_0)\|)$ by making use to Eq.36 and Lemma.2.1.2. Therefore, by Lemma.2.3.1, the equilibrium is uniformly stable.

Consider $B_r \subset D$, and constant c which is less than the minimum of W_1 evaluated at the boundary of B_r: $c < min_{\|x\|=r}W_1(x)$, choosing such c is to make sure the following sets are interior of B_r. Consider $\Omega_{W_1} = \{x \in B_r | W_1(x) \leqslant c\}, \Omega_{t,c} = \{x \in B_r | V(t,x) \leqslant c\}, \Omega_{W_2} = \{x \in B_r | W_2(x) \leqslant c\}$. We have $\Omega_{W_2} \subset \Omega_{W_{t,c}} \subset \Omega_{W_1} \subset B_r \subset D$. Since the derivative of $V(t,x)$ along the trajectory of Eq.30 is nonpositive, solutions started in Ω_{W_2} will stay in $\Omega_{W_{t,c}}$ hence in Ω_{W_1}. Since Lemma.2.1.2 only applied to $x(t)$ that's within a closed ball, the purpose of showing this boundedness result is to make sure Lemma.2.1.2 can be applied to $x(t), \forall t \geqslant t_0$. To show Lemma.2.3.1, STS to bound $\|x(t)\|$ above with a class $\mathscr{K}$ function. By Lemma.2.1.2, there exists class $\mathscr{K}$ functions α_1, α_2 such that

$$\alpha_1(\|x\|) \leqslant V(t,x) \leqslant \alpha_2(\|x\|) \tag{37}$$

Using the fact that $V(t,x)$ is decreasing along the trajectory of Eq.30 and applying both sides by α_1^{-1}, we obtain:

$$\|x(t)\| \leqslant \alpha_1^{-1}(V(t,x(t))) \leqslant \alpha_1^{-1}(V(t_0,x(t_0))) \leqslant \alpha_1^{-1}(\alpha_2(\|x(t_0)\|)) \tag{38}$$

Since the composition of class $\mathscr{K}$ function is still of class $\mathscr{K}$, Lemma.2.3.1 shows the equilibrium is uniformly stable. □

Theorem 2.3.2. *(uniform asymptotic stability) Let x=0 be an equilibrium for Eq.30. Suppose $D \subset \mathbb{R}^n$ is a domain containing $x = 0$. If there exists $V : [0,\infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that*

$$W_1(x) \leqslant V(t,x) \leqslant W_2(x) \tag{39}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leqslant -W_3(x) \tag{40}$$

$\forall t \geqslant 0, \forall x \in D$, where $W_1(x), W_2(x), W_3(x)$ are continuous positive definite functions on D. THEN the equilibrium is uniformly asymptotically stable. If $D = \mathbb{R}^n$ and $W_1(x)$ are radially unbounded, then the equilibrium is globally asymptotically stable.

Note that the constraint on $\dot{V}(t,x)$ is tighter compared with Theorem.2.3.1, which is analogous to requiring a negative definite derivative in the autonomous case. In Theorem.2.3.1, the derivative is allowed to be zero other than $\{0\}$, but now the derivative has to be negative on $D/\{0\}$.

Proof. The proof consists of two parts: 1. boundedness argument making sure Lemma.2.1.2 is applicable. 2. Using Lemma.2.1.2, construct a class $\mathscr{KL}$ function that satisfies Lemma.2.3.1.
Choose $r > 0$, $c > 0$ such that $B_r \in D$, $c < min_{\|x\|=r}W_1(x)$. Consider $\Omega_{W_1} = \{x \in B_r | W_1(x) \leqslant c\}, \Omega_{t,c} = \{x \in B_r | V(t,x) \leqslant c\}, \Omega_{W_2} = \{x \in B_r | W_2(x) \leqslant c\}$. By Eq.39, we have $\Omega_{W_2} \subset \Omega_{W_{t,c}} \subset \Omega_{W_1} \subset B_r \subset D$. Since the derivative of $V(t,x)$ along the trajectory of Eq.30 is negative, solutions started in Ω_{W_2} will stay in $\Omega_{W_1} \subset B_r$. Therefore, there exists $B_\delta \subset \Omega_{W_2}$ such that if $\|x(t_0)\| \in B_\delta$, $\|x(t)\| \in B_r$, which allows us to apply Lemma.2.1.2 to $x(t), \forall t \geqslant t_0$. By Lemma.2.1.2, there exists class $\mathscr{K}$ functions $\alpha_1, \alpha_2, \alpha_3$ such that

$$\dot{V}(t,x) \leqslant -W_3(x) \leqslant -\alpha_3(\|x\|) \tag{41}$$

and

$$\alpha_1(\|x\|) \leqslant V(t,x) \leqslant \alpha_2(\|x\|) \tag{42}$$

By Eq.42, $\alpha_2^{-1}(V) \leqslant \|x\| \iff \alpha_3(\alpha_2^{-1}(V)) \leqslant \alpha_3(\|x\|) \iff -\alpha_3(\|x\|) \leqslant -\alpha_3(\alpha_2^{-1}(V))$. Therefore by Eq.41

$$\dot{V}(t,x) \leqslant -\alpha_3(\alpha_2^{-1}(V)) \tag{43}$$

which is a differential inequality. Note that $\alpha = \alpha_3 \circ \alpha_2^{-1}$ is class $\mathscr{K}$ defined on $[0,r]$. If α is not locally Lipschitz, there exists a locally Lipschitz function β such that $\beta \leqslant \alpha$. Hence, WLOG, we can assume α to be locally Lipschitz. Consider the differential equation based on Eq.43: $\dot{y} = -\alpha(y), y(t_0) = V(t_0, x(t_0)) > 0$. By Lemma.2.1.3, we have $V(t,x(t)) \leqslant y(t), \forall t \geqslant t_0$. By Lemma.2.1.4, there exists class $\mathscr{KL}$ function $\sigma(r,s)$ defined on $[0,r] \times [0,\infty)$ such that $y(t) = \sigma(V(t_0,x(t_0)), t-t_0)$. Hence, $V(t,x(t)) \leqslant \sigma(V(t_0,x(t_0)), t-t_0), \forall V(t_0,x(t_0)) \in [0,c]$. Since by Eq.42,

$$\|x\| \leqslant \alpha_1^{-1}(V(t,x(t))) \leqslant \alpha_1^{-1}(\sigma(V(t_0,x(t_0)),t-t_0)) \leqslant \alpha_1^{-1}(\sigma(\alpha_2(\|x(t_0)\|),t-t_0)) \tag{44}$$

By Lemmq.2.1.1, $\alpha_1^{-1}(\sigma(\alpha_2(\cdot),\cdot))$ is a class $\mathscr{KL}$ function. By Lemma.2.3.1, the equilibrium is uniformly asymptotically stable.
When $D = \mathbb{R}^n$, the condition of W_1 being radially unbounded is needed to make sure the constant $c < min_{\|x\|=r} W_1$ in the proof can be chosen arbitrarily large so that trajectory arbitrarily far away from the equilibrium can be bounded, then approach zero. □

2.4 Boundedness

Before studying systems with "parameters", an important notion is to understand the boundedness of trajectories. For nonautonomous systems:

Definition 2.4.1. The solution of Eq.30 are

- uniformly bounded if there exists a constant $c > 0$(independent of t_0) and for every $a \in (0,c)$, there is $\beta = \beta(a) > 0$(independent of t_0) such that $\|x(t_0)\| \leqslant a \implies \|x(t)\| \leqslant \beta, \forall t \geqslant t_0$.
- globally uniformly bounded if the above inequality holds for any large a
- uniformly ultimately bounded with ultimate bound b if there exist positive constants b, c(independent of $t_0 \geqslant 0$), and for every $a \in (0,c)$, there is $T = T(a,b)$(independent of t_0) such that $\|x(t_0)\| \leqslant a \implies \|x(t)\| \leqslant b, \forall t \geqslant t_0 + T$.
- globally uniformly ultimately bounded if the above inequality holds for arbitrarily large a.

Boundedness is "weaker" compared with ultimate boundedness, because the bound it chose is universal ($t \geqslant t_0$), while the bound in ultimate boundedness is only satisfied if let the system progress for long enough ($t \geqslant t_0 + T$), to capture the variation in the trajectory. For autonomous systems, drop uniformly.
The definitions of boundedness and stability are different. In the definition of stability, given a desired distance away from the equilibrium, one can find where to start the system.

In the definition of boundedness, given an allowed range of the norm of the starting point, one can find the distance from the origin.
Boundedness does not imply stability, and stability does not imply boundedness, they are two different aspects of a dynamical system.
Here is a theorem about showing boundedness we will use in the proof stability:

Theorem 2.4.1. *Let $D \subset \mathbb{R}^n$ be a domain that containss the origin and $V : [0,\infty) \times D \to \mathbb{R}$ be a continously differentiable function such that*

$$\alpha_1(\|x\|) \leqslant V(t,x) \leqslant \alpha_2(\|x\|) \tag{45}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leqslant -W_3(x), \forall \|x\| \geqslant \mu > 0 \tag{46}$$

$\forall t \geqslant 0, \forall x \in D$, where α_1, α_2 are class $\mathscr{K}$ functions and $W_3(x)$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that $\mu < \alpha_2^{-1}(\alpha_1(r))$. Then, there exists a class $\mathscr{K}\mathscr{L}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \leqslant \alpha_2^{-1}(\alpha_1(r))$, there is $T \geqslant 0$(dependent on $x(t_0)$ and μ) such that the solution of Eq.30 satisfies

$$\|x(t)\| \leqslant \beta(\|x(t)\|, t-t_0), \forall t_0 \leqslant t \leqslant t_0 + T \tag{47}$$

$$\|x(t)\| \leqslant \alpha_1^{-1}(\alpha_2(\mu)), \forall t \geqslant t_0 + T \tag{48}$$

Moreover, if $D = \mathbb{R}^n$ and α_1 belongs to class $\mathscr{K}_\infty$, then Eq.47, 48 holds for any initial state $x(t_0)$ with no restriction on how large μ is.

Note that if $\mu = 0$, Theorem.2.4.1 is equivalent to Theorem.2.3.2. The theorem requires a less strict condition where Eq.48 is only needed outside a radius μ, but the result shows the solution is still bounded even after a long enough time.

Proof. Strategy: the proof is the reverse engineer of Pages 169-172[3]. We construct two regions: one is the region that behaves like stability, and another is the region that's ultimately bounded. T turns out to be the finite time that all solutions enter the ultimately bounded region.
Let $\eta = \alpha_2(\mu), \rho = \alpha_1(r)$. Since $\mu < \alpha_2^{-1}(\alpha_1(r))$, $\eta < \rho$. Consider

$$\Omega_{t,\eta} = \{x \in B_r | V(t,x) \leqslant \eta\} \tag{49}$$

$$\Omega_{t,\rho} = \{x \in B_r | V(t,x) \leqslant \rho\} \tag{50}$$

Since $\alpha_1(\|x\|) \leqslant V(t,x)$ and $V(t,x) \leqslant \rho$ implies $\|x\| \leqslant r$, $\Omega_{t,\rho} \subset B_r$. Since $V(t,x) \leqslant \alpha_2(\|x\|)$ and $\|x\| \leqslant \mu$ implies $V(t,x) \leqslant \alpha_2(\mu)$, $B_\mu \subset \Omega_{t,\eta}$. Hence:

$$B_\mu \subset \Omega_{t,\eta} \subset \Omega_{t,\rho} \subset B_r \tag{51}$$

Since for $\|x\| \geqslant \mu$, $\dot{V} < 0$. Since the boundary of $\Omega_{t,\eta}, \Omega_{t,\rho}$ are outside B_μ, all trajectories started in $\Omega_{t,\eta}$ or $\Omega_{t,\rho}$ will not leave since $\dot{V}$ is negative on the boundary. Since $\|x(t_0)\| \leqslant \alpha_2^{-1}(\rho)$ and $V(t_0,x(t_0)) \leqslant \alpha_2(\|x(t_0)\|)$, $V(t_0,x(t_0)) \leqslant \rho$. Hence in this case all t_0 satisfies $x(t_0) \in \Omega_{t,\rho}$.
Consider $\omega = \Omega_{t,\rho} - \Omega_{t,\eta}$. W_3 has a minimum k on ω because W_3 is continous and Ω is compact. Hence, $\dot{V}(t,x) \leqslant -W_3 \leqslant -k$, which means $V(t,x(t)) \leqslant V(t_0,x(t_0)) - k(t-t_0) \leqslant$

$\rho - k(t-t_0)$. Hence, $x(t)$ enters $\Omega_{t,\eta}$ between $[t_0, t_0 + (\rho - \eta)/k]$, all solutions started in $\Omega_{t,\rho}$ will enter $\Omega_{t,\eta}$ in finite amount of time. Let $T = (\rho - \eta)/k$.
When $0z \leqslant t_0 \leqslant t_0 + T$, $x(t) \in \Omega_{t,\eta}$, $\alpha_1(\|x\|) \leqslant V(t,x(t)) \leqslant \eta = \alpha_2(\mu)$, hence $\|x(t)\| \leqslant \alpha_1^{-1}(\alpha_2(\mu))$. When $t_0 \geqslant t_0 + T$, $x(t) \in \Omega_{t,\rho}$, since the solutions are bounded, follows the exact same proof of Theorem.2.3.2, we can show there exists a class $\mathscr{K}\mathscr{L}$ function σ such that

$$V(t,x(t)) \leqslant \sigma(V(t_0,x(t_0)), t-t_0), \forall t \in [t_0, t_0+T] \tag{52}$$

If $D = \mathbb{R}^n$, ρ can be chosen arbitrarily large, mea □

2.5 Input-to-State Stability (ISS)

We add some parameters determined by the input function to the nonautonomous system. Consider the system

$$\dot{x} = f(t,x,u) \tag{53}$$

Suppose $f : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x and u. The input function $u(t)$ is a piecewise continuous, bounded function of t for all $t \geqslant 0$. We can view Eq,53 as a perturbation of the unforced system (when $u = 0$). For instance when f is Lipschitz in the following way: $\|f(t,x,u) - f(t,x,0)\| \leqslant L\|u\|$, this gives $\left\|\dot{V} - \dot{V_0}\right\| \leqslant L\|u\| \left\|\frac{\partial V}{\partial x}\right\|$ where V_0 comes from the unforced system. Since u is bounded, it is possible that $\dot{V}$ is negative outside a radius of μ, where μ depends on $sup\|u\|$.
If $\dot{V} < 0$ outside a ball of radius μ, we can apply Theorem.2.4.1, where $\|x(t)\|$ is bounded by a class $\mathscr{K}\mathscr{L}$ function $\beta(\|x(t_0)\|, t-t_0)$ over $[t_0, t_0+T]$ and by a class $\mathscr{K}$ function $\alpha_1^{-1}(\alpha_2(\mu))$ for $t \geqslant t_0 + T$, which means

$$\|x(t)\| \leqslant \beta(\|x(t_0)\|, t-t_0) + \alpha_1^{-1}(\alpha_2(\mu)), \forall t \geqslant t_0 \tag{54}$$

Therefore, we are motivated to present the definition of input-to-state stability, and the Lyapunov-like theorem that gives sufficient conditions for input-to-state stability.

Definition 2.5.1. The system Eq.53 is said to be input-to-state stable if there exists a class $\mathscr{K}\mathscr{L}$ function β and a class $\mathscr{K}$ function γ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geqslant t_0$ and satisfies

$$\|x(t)\| \leqslant \beta(\|x(t_0)\|, t-t_0) + \gamma(sup_{t_0 \leqslant \tau \leqslant t}\|u(\tau)\|) \tag{55}$$

Theorem 2.5.1. *Let $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continously differentiable function such that*

$$\alpha_1(\|x\|) \leqslant V(t,x) \leqslant \alpha_2(\|x\|) \tag{56}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leqslant 0 \leqslant -W_3(x), \forall \|x\| \geqslant \rho(\|u\|) > 0 \tag{57}$$

$\forall (t,x,u) \in [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$, where α_1, α_2 are class $\mathscr{K}_\infty$ functions, ρ is a class $\mathscr{K}$ function, and $W_3(x)$ is a continuous positive definite function on $\mathbb{R}^n$. Then, the system Eq.53 is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Proof. By the global version of the Theorem.2.3.1, we have:

$$\|x(t)\| \leqslant \beta(\|x(t_0)\|, t-t_0) + \gamma(\mu) \tag{58}$$

where there's no restriction on how large $\mu > 0$ should be, hence we take $\mu = sup_{\tau \geqslant t_0} \|u(\tau)\|, \forall t \geqslant t_0$. Since $t(t)$ depends only on $u(\tau)$ for $t_0 \leqslant \tau \leqslant t$, the supremum on the right-hand side of Eq.58 can be taken over $[t_0, t]$, which yields the definition of input-to-state stability. □

Lemma 2.5.1. *Suppose $f(t,x,u)$ is continuously differentiable and globally Lipschitz in (x,u), uniformly in t. If the unforced system of Eq.53 has a globally exponentially stable equilibrium point at the origin $x = 0$, then Eq.53 is input-to-state stable.*

3 Stability in terms of two measures

Here we introduce a generalization of the stability analysis, where the distance from the equilibrium of the initial time and final time are measured with two different "functions", for instance, it can be Euclidean 2-norm as before, or something more general. This generalization enables us to unify a variety of stability notions. With this generalization, all the inequalities such as Eq.35, Eq.36 need to be generalized. This section is based on [5].

Consider the nonautonomous system Eq.30. We define two special sets of "measures" we will use

Definition 3.0.1. $\Gamma = \{\text{continuous } h : \mathbb{R}_+ \times \mathbb{R} | \inf_{(t,x)} h(t,x) = 0\}$
$\Gamma_0 = \{h \in \Gamma | \inf_x h(t,x) = 0 \text{ for each } t \in \mathbb{R}_+\}$

Definition 3.0.2. Suppose $h, h_0 \in \Gamma$.

(i) h_0 is finer than h if there exists a $\rho > 0$ and a function $\phi \in \mathscr{K}$ such that $h_0(t,x) < \rho$ implies $h(t,x) \leqslant \phi(t, h_0(t,x))$.

(ii) h_0 is uniformly finer than h if in (i) ϕ is independent of t.

(iii) h_0 is asymptotically finer than h if there exists a $\rho > 0$ and a function $\phi \in \mathscr{KL}$ such that $h_0(t,x) < \rho$ implies $h(t,x) \leqslant \phi(h_0(t,x), t)$.

The definition of stability is adjusted using two measures by replacing 2-norm with functions in Γ.

Definition 3.0.3. Suppose $h, h_0 \in \Gamma$. The dynamical system Eq.30 is

(S1) $(h, h_0)-$equi-stable if $\forall\ \varepsilon > 0, t_0 \in \mathbb{R}_+$, $\exists\ \delta(t_0, \varepsilon) > 0$ that is continuous in t_0 such that for all $t \geqslant t_0$, $h_0(t_0, x(t_0)) < \delta$ implies $h(t, x(t)) < \varepsilon$.

(S2) $(h_0, h)-$uniformly stable if δ in S1 is independent of time.

(S3) $(h_0, h)-$equi-attractive if $\forall\ \varepsilon > 0, t_0 \in \mathbb{R}^+$, $\exists\ \delta_0(t_0)$ and $T = T(t_0, \varepsilon)$ such that $h_0(t_0, x_0) < \delta_0$ implies $h(t, x(t)) < \varepsilon, t \geqslant t_0 + T$. (For each $(t_0, \varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^+$, there exists $(\delta(t_0), T(t_0, \varepsilon)) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $h_0(t_0, x_0) < \delta_0$ implies $h(t, x(t)) < \varepsilon, t \geqslant t_0 + T$.)

(S4) $(h_0,h)-$uniformly attractive if δ_0 and T in S3 are independent of time.

(S5) $(h_0,h)-$equi-asymptotically stable if (S1) and (S3) hold simultaneously.

(S6) $(h_0,h)-$uniform-asymptotically stable is (S2) and (S4) holds simultaneously.

(S7) $(h_0,h)-$equi-attractive in the large if $\forall\varepsilon > 0, \alpha > 0$ and $t_0 \in \mathbb{R}^+$, there exists a positive number $T = T(t_0, \varepsilon, \alpha)$ such that $h_0(t_0,x_0) < \alpha$ implies $h(t,x(t)) < \varepsilon, t \geqslant t_0 + T$.

(S8) $(h_0,h)-$uniformly attractive in the large if the constant T in S7 is independent of t_0.

(S9) $(h_0,h)-$unstable if (S1) fails to hold.

Here stability and attractiveness are defined separately. There are systems that are attractive but not stable, meaning even though the system goes to 0 after a long enough time, it cannot stay arbitrarily close to the equilibrium for all time. Attractive in the large is stronger than attractive since in attractive in the large, for any initial state, the solution can go arbitrarily close to the equilibrium.

Definition 3.0.4. Let continuous function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n_+$, then V is said to be

(i) h-positive definite if there exists a $\rho > 0$ and a function $b \in \mathscr{K}$ such that $b(h(t,x)) \leqslant V(t,x)$ whenever $h(t,x) < \rho$.

(i) h-descrescent if there exists a $\rho > 0$ and a function $a \in \mathscr{K}$ such that $V(t,x) \leqslant a(h(t,x))$ whenever $h(t,x) < \rho$.

(ii) h-weakly-decrescent if there exists a $\rho > 0$ and a function $a \in \mathscr{C}\mathscr{K}$ such that $V(t,x) \leqslant a(t,h(t,x))$ whenever $h(t,x) < \rho$.

(iii) h-asymptotically decrescent if there exists a $\rho > 0$ and a function $a \in \mathscr{K}\mathscr{L}$ such that $V(t,x) \leqslant a(h(t,x),t)$ whenever $h(t,x) < \rho$.

To broaden the set of systems we work with, we attempt to include more systems that are not differentiable in the traditional way by generalizing the notion of derivative.

Theorem 3.0.1. *We define the sudo-Dini derivatives of the Lyapunov function $V(t,x(t)) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^N_+]$:*

$$D^+V(t,x(t)) = \lim_{\delta\to 0^+} \sup \frac{1}{\delta}[V(t+\delta, x+\delta f(t,x)) - V(t,x)] \tag{59}$$

$$D_-V(t,x(t)) = \lim_{\delta\to 0^-} \inf \frac{1}{\delta}[V(t+\delta, x+\delta f(t,x)) - V(t,x)] \tag{60}$$

Its Dini derivative is defined as

$$D^+V(t,x(t)) = \lim_{\delta\to 0^+} \sup \frac{1}{\delta}[V(t+\delta, x(t+\delta)) - V(t,x)] \tag{61}$$

$$D_-V(t,x(t)) = \lim_{\delta\to 0^-} \inf \frac{1}{\delta}[V(t+\delta, x(t+\delta)) - V(t,x)] \tag{62}$$

If $V(t,x)$ is locally Lipschitz in x, then the Dini derivative equals to the sudo-Dinni derivative.

Proof. The proof utilizes Taylor's expansion and the definition of Lipschitzian continuity. □

If the Lyapunov function is differentiable, the Dini derivative is the traditional derivative: $D^+V(t,x) = D_-V(t,x) = V'(t,x) = V_t(t,x) + V_x(t,x)f(t,x)$. Dini derivative works well with monotonicity as illustrated in the following lemma:

Lemma 3.0.1. *Suppose $m(t)$ is continuous on (a,b). Then $m(t)$ is nondecreasing (nonincreasing) on (a,b) if and only if $D^+m(t) \geqslant 0 (\leqslant 0)$ for every $t \in (a,b)$, where*

$$D^+m(t) = \lim_{\delta \to 0^+} \sup \frac{1}{\delta}[m(t+\delta) - m(t)] \tag{63}$$

Proof. The forward direction is immediate. Suppose $D^+m(t) \geqslant 0$ on (a,b). For the sake of contradiction, suppose $\alpha, \beta \in (a,b), \alpha < \beta$, such that $m(\alpha) > m(\beta)$. Since m is continuous, by mean value theorem, there exists μ such that $m(\alpha) > \mu > m(\beta)$, which means there is $t \in [\alpha, \beta]$ such that $m(t) > \mu$. Let $\eta = sup\{t | m(t) > \mu, t \in [\alpha, \beta]\}$. Clearly, $\eta \in (\alpha, \beta)$ and $m(\eta) = \mu$. For all $t \in (\eta, \beta)$ and small enough $\delta > 0$, $m(t+\delta) \leqslant m(\eta)$, $\frac{m(t+\delta)-m(t)}{\delta} \leqslant \frac{m(\eta)-m(t)}{\delta} \leqslant 0$, meaning $\frac{m(\eta)-m(t)}{\delta} \leqslant \frac{m(\eta)-m(t)}{\eta - t} < 0$. Therefore, $D^+m(t) < 0$ which is a contradiction. □

4 ISS in terms of two measures

After formalizing the Dini derivative and the comparison result of the Dini derivative, the first step is to generalize Theorem.2.4.1.

Theorem 4.0.1. *Suppose $h, h_0 \in \Gamma$. Let $D \subset \mathbb{R}^n$ be a domain contains the equilibrium of interest and $V : [0, \infty) \times D \to \mathbb{R}$ is continuous, locally Lipschitz in x, and there exists $\alpha_1, \alpha_2, c \in \mathscr{K}$ such that*

$$\alpha_1(h(t,x(t))) \leqslant V(t,x) \leqslant \alpha_2(h_0(t,x(t))) \tag{64}$$

$$D^+V(t,x(t)) \leqslant -c(h_0(t,x)), \text{ on } S^C(h_0, \mu) \tag{65}$$

Take $r > 0$ such that $S(h,r) \subset D$ and suppose that $\mu < \alpha_2^{-1}(\alpha_1(r))$. For every initial state $x(t_0)$, satisfying $h_0(t_0, x(t_0)) \leqslant \alpha_2^{-1}(\alpha_1(r))$, $\exists\, T \geqslant 0$(dependent on $x(t_0)$ and μ), $\exists\, \beta \in \mathscr{KL}$ such that the solution of Eq.30 satisfies

$$h(t,x(t)) \leqslant \beta(h_0(t_0,x_0), t - t_0), \forall t_0 \leqslant t \leqslant t_0 + T \tag{66}$$

$$h(t,x(t)) \leqslant \alpha_1^{-1}(\alpha_2(\mu)), \forall t \geqslant t_0 + T \tag{67}$$

Moreover, if $D = \mathbb{R}^n$ and $\alpha_1 \in \mathscr{K}_\infty$, then Eq.66 and Eq.67 hold for any initial state $x(t_0)$ with no restriction on how large μ is.

Proof. Let $\rho = \alpha_1(r), \eta = \alpha_2(\mu)$, therefore $\eta < \rho$. Consider

$$\Omega_{t,\eta} = \{x \in S(h,r) | V(t,x) \leqslant \eta\} \Omega_{t,\rho} = \{x \in S(h,r) | V(t,x) \leqslant \rho\} \tag{68}$$

Because of Eq.99, we have:

$$S(h_0,\mu) \subset \Omega_{t,\eta} \subset S(h, \alpha_1^{-1}(\eta)) \subset S(h, \alpha_1^{-1}(\rho)) = S(h,r) \subset D \tag{69}$$

$$\Omega_{t,\eta} \subset \Omega_{t,\rho} \subset S(h, \alpha_1^{-1}(\rho)) = S(h,r) \subset D \tag{70}$$

which suggests that $\Omega_{t,\rho} - \Omega_{t,\eta} \subset S^C(h_0,\mu)$.
Note that all solutions started in either $\Omega_{t,\rho}$ or $\Omega_{t,\eta}$ won't leave because $D^+V(t,x)$ is negative on the boundaries of both sets. Since $\alpha_2(h_0(t_0,x(t_0)) \leqslant \rho$, we have $x(t_0) \in \Omega_{t_0,\rho}$. Therefore, we have $x(t) \in \Omega_{t,\rho}.\forall t \geqslant t_0$. Since $D^+V(t,x) \leqslant -c(h_0(t,x))$ on $S^C(h_0,\mu)$, and since $c \in \mathscr{K}$, we have $D^+V(t,x) \leqslant -c(h_0(t,x)) \leqslant -c(\mu) \equiv -K$ over the set $S^C(h_0,\mu) \cap S(h,r)$, which contains $\Omega_{t,\rho} - \Omega_{t,\eta}$. Therefore, we have $\frac{V(t,x(t))-V(t_0,x(t_0))}{t-t_0} \leqslant -K$, hence $V(t,x(t)) \leqslant V(t_0,x(t_0)) - K(t-t_0) \leqslant \rho - K(t-t_0)$, which shows that $V(t,x(t))$ reduces to η within $[t_0, t_0 + \frac{\rho-\eta}{k}]$. This tells us that trajectories entered $\Omega_{t,\rho}$ will enter $\Omega_{t,\eta}$ at, say, $t_0 + T$, which satisfies $t_0 + T \leqslant t_0 + \frac{\rho-\eta}{k}$.
For solutions entered $\Omega_{t,\eta}$, since $\Omega_{t,\eta} \subset S(h, \alpha_1^{-1}(\eta))$, we have that $\forall t \geqslant t_0 + T$, $h(t,x(t)) \leqslant \alpha_1^{-1}(\eta) = \alpha_1^{-1}(\alpha_2(\mu))$.
For solutions inside $\Omega_{t,\rho}$ but outside $\Omega_{t,\eta}$, aka $\forall t \in [t_0, t_0 + T]$, $D^+V(t,x(t)) \leqslant -c(h_0(t,x(t))) \leqslant -c(\alpha_2^{-1}(V(t,x(t)))) \equiv -\alpha(V(t,x(t)))$, $\alpha \in \mathscr{K}$, which corresponds to the ode $\dot{y} = -\alpha(y), y(t_0) = V(t_0,x(t_0))$. WLOG we can assume α is Lipschitz (if not we can choose a function bounded below by α that's Lipschitz). By Lemma.2.1.3, $V(t,x(t)) \leqslant y(t), \forall t \geqslant t_0$. By Lemma.2.1.4, there exists $\sigma \in \mathscr{KL}$ such that $y(t) = \sigma(y_0, t-t_0)$, which yields $V(t,x(t)) \leqslant \sigma(V(t_0,x(t_0)), t-t_0)$. Therefore, $h(t,x) \leqslant \alpha_1^{-1}(\sigma(V(t_0,x(t_0)), t-t_0)) \leqslant \alpha_1(\sigma(\alpha_2(h_0(t_0,x(t_0))), t-t_0))) = \beta(h_0(t_0,x_0), t-t_0), \forall t \in [t_0, t_0+T]$
If $\alpha_1 \in \mathscr{K}_\infty$, therefore $\alpha_2 \in \mathscr{K}_\infty$. If we also know that $D = \mathbb{R}^n$, for any arbitrary μ and $h_0(t_0,x_0)$ given, r for B_r can be chosen arbitrarily large, making sure that $\mu < \alpha_2^{-1}(\alpha_1(r))$ and $h_0(t_0,x(t_0)) \leqslant \alpha_2^{-1}(\alpha_1(r))$, and the proof will go exactly the same as above. □

Definition 4.0.1. The system Eq.53 is Input-to-state stable in terms of two measures if $\exists \beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists $\forall t \geqslant t_0$ and satisfies

$$h(t,x) \leqslant \beta(h_0(t_0,x_0), t-t_0) + \gamma(sup_{t_0 \leqslant \tau \leqslant t} \|u(\tau)\|) \tag{71}$$

Theorem 4.0.2. *Suppose $h, h_0 \in \Gamma$. Let $D \subset \mathbb{R}^n$ be a domain contains the equilibrium of interest and $V : [0,\infty) \times D \to \mathbb{R}$ is continuous, locally Lipschitz in x, and there exists $\alpha_1, \alpha_2, c, \rho \in \mathscr{K}$ such that*

$$\alpha_1(h(t,x(t))) \leqslant V(t,x) \leqslant \alpha_2(h_0(t,x(t))) \tag{72}$$

$$D^+V(t,x(t)) \leqslant -c(h_0(t,x)),\ on\ S^C(h_0, \rho(\|u\|)) \tag{73}$$

Then, Eq.53 is $(h_0,h)-$input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Proof. By Theorem,5.0.4, $\forall t \geqslant t_0$

$$h(t,x) \leqslant \beta(h_0(t_0,x(t_0))) + \alpha_1^{-1}(\alpha_2(\rho(\|u\|))) \tag{74}$$
$$\leqslant \beta(h_0(t_0,x(t_0))) + \gamma(sup_{\tau \geqslant t_0} \|u(\tau)\|) \tag{75}$$
$$\leqslant \beta(h_0(t_0,x(t_0))) + \gamma(sup_{t \geqslant \tau \geqslant t_0} \|u(\tau)\|) \tag{76}$$

the last step is because $x(t)$ is only related to $u(t)$ for $t \in [t_0,t]$. □

5 ISS of Dynamic systems on time scales in terms of two measures

For real-life applications, we extend the notion of input-to-state stability to dynamic systems on time scale.

Theorem 5.0.1. *Suppose $f : \mathbb{T} \to \mathbb{R}$ is differentiable and $f^\Delta \geqslant 0$($f^\Delta \leqslant 0$) on $[a,b] \subset \mathbb{T}$. Then f is non-decreasing(non-increasing).*

Proof. FOSC, suppose $\alpha, \beta \in [a,b]$ such that $\alpha < \beta, f(\alpha) > f(\beta)$. First we consider a special, case, when $[\alpha,\beta] = \{\alpha,\beta\}$, i.e., Suppose α is right scattered β is left-scattered and α,β are consecutive. By definition of f^Δ, $f^\Delta(\alpha) < 0$, which is a contradiction.
Suppose $\exists \mu \in [\alpha,\beta]$ such that $f(\alpha) > \mu > f(\beta)$. Let $\eta = sup\{t | f(t) > \mu\}$, which means $\forall t \in (\eta,\beta), f(t) \leqslant \mu$.

- If η is right-scattered, $f(\eta) > \mu$. $\forall t \in (\eta,\beta), \frac{f(t)-f(\eta)}{t-\eta} < 0$. Since $\sigma(\eta) \in (\eta,\beta)$, we have $\frac{f(\sigma)-f(\eta)}{\mu(\eta)} = f^\Delta(\eta) < 0$, which contradicts $f^\Delta \geqslant 0$.
- If η is right-dense, $f(\eta) = \mu$, and the proof goes the same as the usual calculus. $\forall t \in (\eta,\beta)$ and $\forall \delta > 0$ small enough which guarantees $\frac{f(t)-f(\eta)}{\delta} < \frac{f(t)-f(\eta)}{t-\eta} \leqslant 0$, which implies $\frac{f(\eta+\delta)-f(\eta)}{t-\eta} < 0$, and after taking limit $\delta \to 0^+$ yields $f^\Delta < 0$ which is a contradiction.

□

The converse of Theorem.5 is in [1]. To generalize Theorem.5.0.4 to time scale, we need to define the Dini derivative in terms of time scale calculus and the comparison result. Consider the dynamic system

$$x^\Delta = f(t,x), x(t_0) = x_0 \tag{77}$$

where $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, and x^Δ denotes the derivative of x with respect to $t \in \mathbb{T}$.

Definition 5.0.1. [4] Let $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$. Then we define the generalized derivative of $V(t,x)$ relative to Eq.77 as follows: given $\varepsilon > 0$, there exists a neighbourhood $N(\varepsilon)$ of $t \in \mathbb{T}$ such that

$$\frac{1}{\mu(s,t)}[V(\sigma(t),x(\sigma(t))) - V(s,x(\sigma(t)) - \mu(s,t)f(t,x(t)))] < D^+V^\Delta(t,x(t)) + \varepsilon \tag{78}$$

for each $s \in N(\varepsilon)$ and $s > t$, where $\mu(t,s) = \sigma(t) - s$ and $x(t)$ is any solution of Eq.77.

In case, $t \in \mathbb{T}$ is right-scattered and $V(t,x(t))$ is continuous at t, we have

$$D^+V(t,x(t)) = \frac{1}{\mu^*(t)}[V(\sigma(t),x(\sigma(t) - V(t,x(t))))] \tag{79}$$

where $\mu^*(t) = \mu(t,t)$, which is exactly the time scale derivative. When $t \in \mathbb{T}$ is right-dense and $V(t,x(t))$ is continuous at t, $D^+V(t,x(t))$ is the same as the Dini derivative. Since we have the monotonicity property of both the time scale derivative (Theorem.5.0.1) and Dini derivative (Lemma.3.0.1), the monotony property of the derivative defined in Definition.5.0.1 follows.

Theorem 5.0.2. *(Comparison theorem on time scale)[4] Let $V \in C[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $V(t,x)$ be locally Lipschitzian in x for each $t \in \mathbb{T}$ and let*

$$D^+V^\Delta(t,x(t)) \leqslant g(t,V(t,x)) \tag{80}$$

where $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$, $g(t,u)\mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$ and $r(t) = r(t,t_0,u_0)$ is the maximal solution of $u^\Delta = g(t,u), u(t_0) = u_0 \geqslant 0$, existing on $\mathbb{T}$. Then $V(t_0,x_0) \leqslant u_0$ implies that $V(t,x(t)) \leqslant r(t,t_0,u_0), t \in \mathbb{T}, t \geqslant t_0$.

We define special functions on time scales:

Definition 5.0.2. [4] $\mathscr{K} = \{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+] : \sigma(u)$ is strictly increasing in u and $\sigma(0) = 0\}$
$\mathscr{L} = \{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+] : \sigma(u)$ is strictly decreasing in u and $\lim_{u\to\infty} \sigma(u) = 0\}$
$\Gamma = \{h \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+] : inf_{(t,x)} h(t,x) = 0\}$
$\mathscr{CK}$ and $\mathscr{KL}$ are defined the same way as before.

Theorem 5.0.3. *(substitution rule on time scale)[1] Suppose $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\bar{\mathbb{T}} = v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and v is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)v^\Delta(t)\,\Delta t = \int_{v(a)}^{v(b)} f \circ v^{-1}(s)\,\bar{\Delta}s \tag{81}$$

Lemma 5.0.1. *Consider the differential equation on time scale $\mathbb{T}$:*

$$y^\Delta = -\alpha(y), y(t_0) = y_0 \tag{82}$$

where α is a locally Lipschitz class $\mathscr{K}$ function define on $[0,T]$. For all $0 \leqslant y_0 \leqslant T$, this equation has a unique solution $y(t)$ defined for all $t \geqslant t_0$. Moreover,

$$y(t) = \sigma(y_0, t - t_0) \tag{83}$$

where σ is a class $\mathscr{KL}$ function defined on $[0,T] \times [0,\infty]$

Proof. Theorem 2.1.1 of [4] gives the local existence of the solution to IVP at all $t \geqslant t_0$. Since the only solution to $u^\Delta = ku, k \in \mathbb{R}, u(t_0) = 0$ is $u = 0$, and since α is locally Lipschitz, Theorem 2.1.2 of [4] gives the local uniqueness of the solution to the IVP at all $t \geqslant t_0$. Hence, we have the global existence and uniqueness of solutions to the IVP.

By integration, we have

$$\int_{t_0}^{t} -y^{\Delta}\frac{1}{\alpha(y)}\Delta\tau = \int_{t_0}^{t}\Delta\tau \tag{84}$$

Since $y^{\Delta} < 0$ whenever $y > 0$ and $y_0 \geqslant 0$, the solution $y(t)$ is either strictly decreasing or constant 0, hence $-y(\mathbb{T})$ is a time scale. Let $v = -y(t), f(t) = \alpha(y(t))$, since $\alpha \in \mathscr{K}$, by the rule of substitution on time scale (Theorem.5.0.3), we have:

$$\int_{-y(t_0)}^{-y(t)} \frac{1}{\alpha(-s)}\bar{\Delta}s = \int_{t_0}^{t}\Delta\tau \tag{85}$$

Let $-T < b < 0$, define

$$\eta(y) = \int_{b}^{-y} \frac{1}{\alpha(-s)}\bar{\Delta}s \tag{86}$$

Note that $\eta(y) - \eta(y_0) = t - t_0$, therefore we have

$$y(t) = \sigma(y_0, t - t_0) = \eta^{-1}(\eta(y_0) + t - t_0) \tag{87}$$

when $y_0 > 0$, and $y(t) = 0$ when $y_0 = 0$. WTS $\sigma(y_0, t - t_0)$ is in class $\mathscr{K}\mathscr{L}$. STS 1. $\eta(y)$ is strictly decreasing. 2. $\lim_{y\to 0}\eta(y) = \infty$.
1. Let $y_1 > y_2, -y_1, -y_2 \in [0, T]$, $\eta(y_1) - \eta(y_2) = \int_{-y_2}^{-y_1} \frac{1}{\alpha(-s)}\bar{\Delta}s < 0$.
2. Since $y(t)$ is strictly decreasing, it definitely approach 0 from above. But it can only approach 0 asymptotically, not in finite time. It cannot go below zero, since α is not defined for negative inputs, otherwise, the global existence of the solution would be violated. If $y(t)$ goes to 0 in finite time t', then the solution would stay at 0 afterwards due to the 0 derivative, which means around the neighbourhood of t' the solution is not unique (there are at least two: the $y(t)$ that goes to 0 in finite time and the trivial solution). Hence, we have $y(t)$ approaches 0 from above asymptotically. From Eq.85, take limit $t \to \infty$ on both sides, since $y \to 0$ as $t \to \infty$, we have $\lim_{y\to 0}\eta(y) = \infty$.
From 1, since η is strictly decreasing, η^{-1} is also strictly decreasing. For a fixed $t - t_0$, $\sigma(y_0, t - t_0)$ is increasing since the composition of two strictly decreasing functions is increasing. For a fixed y_0, we have $\lim_{t\to\infty}\eta^{-1}(\eta(y_0) + t - t_0) = 0$ from 2. Hence $\sigma(y_0, t - t_0) \in \mathscr{K}\mathscr{L}$ as desired.

□

Theorem 5.0.4. *Suppose $h, h_0 \in \Gamma$. Let $D \subset \mathbb{R}^n$ be a domain contains the equilibrium of interest and $V : \mathbb{T} \times D \to \mathbb{R}_+$ is continuous, locally Lipschitz in x, and there exists $\alpha_1, \alpha_2, c \in \mathscr{K}$ such that*

$$\alpha_1(h(t, x(t))) \leqslant V(t, x) \leqslant \alpha_2(h_0(t, x(t))) \tag{88}$$

$$D^+V^{\Delta}(t, x(t)) \leqslant -c(h_0(t, x)),\ on\ S^C(h_0, \mu) \tag{89}$$

Take $r > 0$ such that $S(h, r) \subset D$ and suppose that $\mu < \alpha_2^{-1}(\alpha_1(r))$.
Moreover, $c(\alpha_2^{-1}(u))\mu^*(t) + u$ is nondecreasing in u $\forall t \in \mathbb{T}$ and $r(t) = r(t, t_0, u_0)$ is the maximal solution of $u^{\Delta} = c(\alpha_2^{-1}(u)), u(t_0) = u_0$ existing on $\mathbb{T}$. (Making sure Theorem.5.0.2 is applicable)

Then for every initial state $x(t_0)$, satisfying $h_0(t_0,x(t_0)) \leqslant \alpha_2^{-1}(\alpha_1(r))$, $\exists\ t_0+T \in \mathbb{T}$(dependent on $x(t_0)$ and μ), $\exists\ \beta \in \mathscr{KL}$ such that the solution of Eq.77 satisfies

$$h(t,x(t)) \leqslant \beta(h_0(t_0,x_0),t-t_0), \forall t_0 \leqslant t \leqslant t_0+T \tag{90}$$

$$h(t,x(t)) \leqslant \alpha_1^{-1}(\alpha_2(\mu)), \forall t \geqslant t_0+T \tag{91}$$

Moreover, if $D=\mathbb{R}^n$ and $\alpha_1 \in \mathscr{K}_\infty$, then Eq.90 and Eq.91 hold for any initial state $x(t_0)$ with no restriction on how large μ is.

Proof. Let $\rho = \alpha_1(r), \eta = \alpha_2(\mu)$, therefore $\eta < \rho$. Consider

$$\Omega_{t,\eta} = \{x \in S(h,r) | V(t,x) \leqslant \eta\}, \Omega_{t,\rho} = \{x \in S(h,r) | V(t,x) \leqslant \rho\} \tag{92}$$

Because of Eq.99, we have:

$$S(h_0,\mu) \subset \Omega_{t,\eta} \subset S(h,\alpha_1^{-1}(\eta)) \subset S(h,\alpha_1^{-1}(\rho)) = S(h,r) \subset D \tag{93}$$

$$\Omega_{t,\eta} \subset \Omega_{t,\rho} \subset S(h,\alpha_1^{-1}(\rho)) = S(h,r) \subset D \tag{94}$$

which suggests that $\Omega_{t,\rho} - \Omega_{t,\eta} \subset S^C(h_0,\mu)$.
Note that all solutions started in either $\Omega_{t,\rho}$ or $\Omega_{t,\eta}$ won't leave because $D^+V^\Delta(t,x)$ is negative on the boundaries of both sets. Since $\alpha_2(h_0(t_0,x(t_0)) \leqslant \rho$, we have $x(t_0) \in \Omega_{t_0,\rho}$. Moreover, we have $x(t) \in \Omega_{t,\rho}.\forall t \geqslant t_0$. Therefore, all the trajectories start in $\Omega_{t,\rho}$ and remain in $\Omega_{t,\rho}$. Since $D^+V^\Delta(t,x) \leqslant -c(h_0(t,x))$ on $S^C(h_0,\mu)$, and since $c \in \mathscr{K}$, we have $D^+V^\Delta(t,x) \leqslant -c(h_0(t,x)) \leqslant -c(\mu) \equiv -K$ over the set $S^C(h_0,\mu) \cap S(h,r)$, which contains $\Omega_{t,\rho} - \Omega_{t,\eta}$. Therefore, we have $\frac{V(t,x(t))-V(t_0,x(t_0))}{t-t_0} \leqslant -K$, hence $V(t,x(t)) \leqslant V(t_0,x(t_0)) - K(t-t_0) \leqslant \rho - K(t-t_0)$, which shows that $V(t,x(t))$ reduces to η within $[t_0, t_0 + \frac{\rho-\eta}{k}]$. This tells us that trajectories entered $\Omega_{t,\rho}$ will enter $\Omega_{t,\eta}$ at, say, t_0+T, which satisfies $t_0+T \leqslant t_0 + \frac{\rho-\eta}{k}, t_0+T \in \mathbb{T}$.
For solutions entered $\Omega_{t,\eta}$, since $\Omega_{t,\eta} \subset S(h,\alpha_1^{-1}(\eta))$, we have that $\forall t \geqslant t_0+T$, $h(t,x(t)) \leqslant \alpha_1^{-1}(\eta) = \alpha_1^{-1}(\alpha_2(\mu))$. For solutions inside $\Omega_{t,\rho}$ but outside $\Omega_{t,\eta}$, aka $\forall t \in [t_0,t_0+T]$, $D^+V^\Delta(t,x(t)) \leqslant -c(h_0(t,x(t))) \leqslant -c(\alpha_2^{-1}(V(t,x(t)))) \equiv -\alpha(V(t,x(t)))$, $\alpha \in \mathscr{K}$, which corresponds to the ode $y^\Delta = -\alpha(y), y(t_0) = V(t_0,x(t_0))$. By Theorem.5.0.2, $V(t,x(t)) \leqslant y(t), \forall t \geqslant t_0, t \in \mathbb{T}$. By Lemma.5.0.1, we have that there exists $\Sigma(y_0,t-t_0) \in \mathscr{KL}$ such that $y(t) = \Sigma(y_0,t-t_0)$. Therefore, there exists $\Sigma \in \mathscr{KL}$ such that $V(t,x(x)) \leqslant \Sigma(y_0,t-t_0)$. Since $y_0 = V(t_0,x(t_0)) \leqslant \alpha_2(h_0(t_0,x(t_0)))$ and $\alpha_1(h(t,x(t)) \leqslant V(t,x)$, we have that

$$h(t,x) \leqslant \alpha_1^{-1}(V(t,x)) \leqslant \alpha_1^{-1}(\Sigma(y_0,t-t_0)) \tag{95}$$

$$\leqslant \alpha_1^{-1}(\Sigma(\alpha_2(h_0(t_0,x(t_0))),t-t_0)) = \sigma(h_0(t_0,x(t_0)),t-t_0) \tag{96}$$

where $\sigma \in \mathscr{KL}$. Hence, we have there exists $\beta \in \mathscr{KL}$ and $T > 0, t_0+T \in \mathbb{T}$ such that

$$h(t,x(t)) \leqslant \beta(h_0(t_0,x_0),t-t_0), \forall t_0 \leqslant t \leqslant t_0+T \tag{97}$$

$$h(t,x(t)) \leqslant \alpha_1^{-1}(\alpha_2(\mu)), \forall t \geqslant t_0+T \tag{98}$$

If $\alpha_1 \in \mathscr{K}_\infty$, therefore $\alpha_2 \in \mathscr{K}_\infty$. If we also know that $D=\mathbb{R}^n$, for any arbitrary μ and $h_0(t_0,x_0)$ given, r for B_r can be chosen arbitrarily large, making sure that $\mu < \alpha_2^{-1}(\alpha_1(r))$ and $h_0(t_0,x(t_0)) \leqslant \alpha_2^{-1}(\alpha_1(r))$, and the proof will go exactly the same as above. □

With Theorem.5.0.4 proven, the ISS on time scale can be easily formulated based on the continuous time case:

Theorem 5.0.5. *Suppose $h, h_0 \in \Gamma$. Let $D \subset \mathbb{R}^n$ be a domain contains the equilibrium of interest and $V : \mathbb{T} \times D \to \mathbb{R}$ is continuous, locally Lipschitz in x, and there exists $\alpha_1, \alpha_2, c, \rho \in \mathscr{K}$ such that*

$$\alpha_1(h(t,x(t))) \leqslant V(t,x) \leqslant \alpha_2(h_0(t,x(t))) \tag{99}$$

$$D^+V^\Delta(t,x(t)) \leqslant -c(h_0(t,x)),\ \text{on } S^C(h_0, \rho(\|u\|)) \tag{100}$$

Moreover, $c(\alpha_2^{-1}(u))\mu^*(t) + u$ is nondecreasing in u $\forall t \in \mathbb{T}$ and $r(t) = r(t,t_0,u_0)$ is the maximal solution of $u^\Delta = c(\alpha_2^{-1}(u)), u(t_0) = u_0$ existing on $\mathbb{T}$. Then, Eq.53 is $(h_0,h)-$input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

6 Conclusion

In this project, we extend the notion of ISS to systems in terms of two measures, which generalized ISS to include many different scenarios such as partial stability and much more. Utilizing time-scale calculus and measure chain theory, we also develop the time-scale counterpart of this extension, which is more applicable to real-life systems. Future research regarding the extension of stability analysis to systems in terms of two measures will enhance the applicability and generality of stability analysis.
ISS in terms of two measures can potentially be applied to event-triggered control systems. Consider the control system on an arbitrary time scale $\mathbb{T}$ with bounded graininess function:

$$z^\Delta = \varepsilon(z(t),u(t)), z(t_0) = z_0, \forall t \in \mathbb{T} \tag{101}$$

where ε is rd-continuous and the control input $u(t)$ is only updated at the discrete time instant $t_i \in \mathbb{T}, i \in \mathbb{N}$, i.e., when the triggering event happens. [2] gives a triggering mechanism with Euclidean norm to maintain the ISS of the system, but they might have used the comparison lemma incorrectly. Moreover, we can potentially extend the triggering mechanism to the two-measure case, and discuss the restrictions on the measures to avoid Zeno behaviour.

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